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bridge, 1707, page 122, where sin AEC is substituted for $\cos \frac{1}{2}(B - C)$ [E being the point, corresponding to D in Professor Lovitt's discussion, when AD is drawn bisecting the angle A.]

Simpson's formulæ were given by Mollweide, without reference to Simpson, in Zach's *Monatliche Correspondenz*, Gotha, volume 18, 1808, page 396.

To various geometrical proofs of the Law of Tangents already indicated in this MONTHLY (*1920*, 53–54, 465–467; *1921*, 71, 79, 170–171), might be added: one by Vignal in *Nouvelles Annales de Mathématiques*, volume 3, 1844, pages 456–457; and one by John Keill, the earliest I have met with, given in his anonymously published *Trigonometriæ Planæ & Sphæricæ*, Oxford, 1715, pp. 16–17.

IV. SOME FORMULAS OF ELEMENTARY TRIGONOMETRY.

BY W. J. RUSK, Grinnell College.

The formulas that are taken for granted are the sine formulas:

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} = \frac{1}{2R},$$

and the projection formulas:

$$a = b\cos\gamma + c\cos\beta, \qquad b = c\cos\alpha + a\cos\gamma, \qquad c = a\cos\beta + b\cos\alpha.$$

If we multiply these in order by a, b, c, and then add the first two results and subtract the third we get one of the cosine formulas; so we shall consider them as given also.

Consider the triangle ABC with $b < a$; take D on AB so that $CA = CD$; then $\angle DCB = \alpha - \beta$ and

$$DB = a\cos\beta - b\cos\alpha = \frac{a^2 - b^2}{c}.$$

1. **Formulas for sin $(\alpha + \beta)$ and sin $(\alpha - \beta)$.** We have from triangle ABC,

$$\frac{\sin\gamma}{c} = \frac{\sin(\alpha+\beta)}{a\cos\beta + b\cos\alpha} = \frac{1}{2R};$$

$$\therefore \quad \sin(\alpha+\beta) = \frac{a}{2R}\cos\beta + \frac{b}{2R}\cos\alpha$$
$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

and from triangle CDB,

$$\frac{\sin\beta}{b} = \frac{\sin(\alpha-\beta)}{a\cos\beta - b\cos\alpha} = \frac{1}{2R},$$

or

$$\sin(\alpha-\beta) = \frac{a}{2R}\cos\beta - \frac{b}{2R}\cos\alpha,$$
$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

Again

$$\begin{aligned}\sin\gamma &= \sin(\alpha+\beta) = (c/a)\sin\alpha\\ &= [(a\cos\beta + b\cos\alpha)\sin\alpha]/a\\ &= \sin\alpha\cos\beta + (b/a)\sin\alpha\cos\alpha\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta.\end{aligned}$$

Also

$$\begin{aligned}1 &= (a/c)\cos\beta + (b/c)\cos\alpha\\ &= (\sin\alpha\cos\beta + \cos\alpha\sin\beta)/\sin\gamma.\end{aligned}$$

$$\therefore\quad \sin\gamma = \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

and from triangle *CDB*,

$$\frac{\sin(\alpha-\beta)}{a\cos\beta - b\cos\alpha} = \frac{\sin\alpha}{a},$$

whence $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

2. **Formulas for cos** $(\alpha+\beta)$ **and cos** $(\alpha-\beta)$. We have from

$$a = b\cos\gamma + c\cos\beta,$$

$$\begin{aligned}-\cos\gamma &= \frac{c}{b}\cos\beta - \frac{a}{b}\\ &= \frac{(a\cos\beta + b\cos\alpha)\cos\beta}{b} - \frac{a}{b}.\\ &= \cos\alpha\cos\beta - \frac{a}{b}(1-\cos^2\beta)\end{aligned}$$

or $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$; and from the triangle *CDB*,

$$a = b\cos(\alpha-\beta) + (a\cos\beta - b\cos\alpha)\cos\beta;$$

$$\begin{aligned}\therefore\quad \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \frac{a}{b}(1-\cos^2\beta)\\ &= \cos\alpha\cos\beta + \sin\alpha\sin\beta.\end{aligned}$$

Again from the triangle *ABC*, we have

$$c^2 = (a\cos\beta + b\cos\alpha)^2 = a^2 + b^2 - 2ab\cos\gamma$$

or

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta + \frac{a}{2b}(\cos^2\beta - 1) + \frac{b}{2a}(\cos^2\alpha - 1)\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta.\end{aligned}$$

And from triangle *CDB*, we find $\cos(\alpha-\beta)$, since

$$(a\cos\beta - b\cos\alpha)^2 = a^2 + b^2 - 2ab\cos(\alpha-\beta).$$

3. **Formulas for tan** $(\alpha+\beta)$ **and tan** $(\alpha-\beta)$. Drop the perpendicular *CE* upon *AB* in the triangle *ABC*. Then

$$\tan\alpha = -\tan(\beta+\gamma) = \frac{EC}{AE} = \frac{(b\cos\gamma + c\cos\beta)\sin\beta}{c - (b\cos\gamma + c\cos\beta)\cos\beta},$$

$$= \frac{\sin\beta\cos\gamma + \frac{c}{b}\cos\beta\sin\beta}{-\cos\beta\cos\gamma + \frac{c}{b}(1-\cos^2\beta)};$$

or,

$$\tan(\beta+\gamma) = \frac{\tan\beta + \tan\gamma}{1 - \tan\beta\tan\gamma}.$$

For tan $(\alpha - \beta)$ drop DF perpendicular to BC. Then

$$\tan(\alpha-\beta) = \frac{FD}{CF} = \frac{(a\cos\beta - b\cos\alpha)\sin\beta}{a - (a\cos\beta - b\cos\alpha)\cos\beta},$$

$$= \frac{\frac{a}{b}\cos\beta\sin\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta + \frac{a}{b}(1-\cos^2\beta)},$$

$$= \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}.$$

Proofs could have been obtained in the same way for the corresponding sine and cosine formulas.

4. **Half-angle formulas.** Produce the side AC to K so that $CK = a$ and apply the cosine formula to the triangle ABK. We have

$$c^2 = (a+b)^2 + 4a^2\cos^2\frac{\gamma}{2} - 2(a+b)2a\cos^2\frac{\gamma}{2}$$

$$= (a+b)^2 - 4ab\cos^2\frac{\gamma}{2}.$$

whence

$$\cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}, \quad \text{and} \quad \sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}.$$

Now consider the triangle CDB. Produce DC to L so that $CL = a$, and apply the cosine formula to the triangle DBL. We have

$$\left(\frac{a^2-b^2}{c}\right)^2 = (a+b)^2 + 4a^2\cos^2\frac{\alpha-\beta}{2} - 2(a+b)2a\cos^2\frac{\alpha-\beta}{2}$$

$$= (a+b)^2 - 4ab\cos^2\frac{\alpha-\beta}{2}$$

or

$$4ab\cos^2\frac{\alpha-\beta}{2} = \left(\frac{a+b}{c}\right)^2[c^2 - (a-b)^2]$$

$$\therefore \quad \cos\frac{\alpha-\beta}{2} = \frac{a+b}{c}\sin\frac{\gamma}{2}; \qquad \sin\frac{\alpha-\beta}{2} = \frac{a-b}{c}\cos\frac{\gamma}{2},$$

and

$$\tan\frac{\alpha-\beta}{2}=\frac{a-b}{a+b}\cot\frac{\gamma}{2}.$$

Again if we call the perimeter of the triangle *CDB*, $2s_1$ and the sides a, b, c_1, it can be very easily proved that

$$s_1=\frac{a+b}{c}(s-b),\qquad s_1-a=\frac{a-b}{c}(s-c),$$

$$s_1-b=\frac{a-b}{c}s,\qquad s_1-c_1=\frac{a+b}{c}(s-a).$$

$$\therefore\quad \sin\frac{\alpha-\beta}{2}=\sqrt{\frac{(s_1-a)(s_1-b)}{ab}}=\frac{a-b}{c}\sqrt{\frac{s(s-c)}{ab}}=\frac{a-b}{c}\cos\frac{\gamma}{2},$$

and

$$\cos\frac{\alpha-\beta}{2}=\sqrt{\frac{s_1(s_1-c_1)}{ab}}=\frac{a+b}{c}\sin\frac{\gamma}{2};$$

which shows that Simpson's formulas and the tangent formula are equivalent to the half-angle formulas.

All of the above formulas have been proved on the condition that the angles are the angles of a triangle but they can be generalized in any of the usual ways.

Again from

$$a=b\cos\gamma+c\cos\beta,\qquad b=a\cos\gamma+c\cos\alpha$$

we have

$$\frac{a-b}{c}=\frac{\cos\beta-\cos\alpha}{1+\cos\gamma},\qquad\text{and}\qquad\frac{a+b}{c}=\frac{\cos\beta+\cos\alpha}{1-\cos\gamma}.$$

$$\therefore\quad\frac{a-b}{c}=\frac{\sin^2\frac{\alpha}{2}-\sin^2\frac{\beta}{2}}{\cos^2\frac{\gamma}{2}}=\frac{\sin^2\frac{\alpha}{2}\left(1-\sin^2\frac{\beta}{2}\right)-\sin^2\frac{\beta}{2}\left(1-\sin^2\frac{\alpha}{2}\right)}{\cos^2\frac{\gamma}{2}},$$

$$=\frac{\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}{\cos^2\frac{\gamma}{2}}=\frac{\sin\frac{1}{2}(\alpha-\beta)}{\cos\frac{\gamma}{2}}$$

$$=\frac{2\sin\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\alpha+\beta)}{\sin\gamma}.$$

But

$$\frac{a-b}{c}=\frac{\sin\alpha-\sin\beta}{\sin\gamma}.$$

$$\therefore\quad\sin\alpha-\sin\beta=2\cos\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta).$$

It is not necessary to go through the detailed work in the case of other analogous formulas.